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Dynamics of damped coupled oscillators near resonance

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We study the dynamics of two conservative oscillators with perturbations from a linear displacement coupling and non-Hamiltonian forces such as damping. We examine the dynamics of these systems when they are near the primary resonance using secular perturbation theory. We show that near resonance a large class of driven oscillators and two coupled oscillators can be transformed to the same ordinary differential equations (ODEs). This common type of dynamics near the resonance is a generalization of the standard Hamiltonian dynamics of two coupled conservative oscillators. We derive expressions for the parameters in these ODEs. From these parameters, we derive analytical expressions for the linear fixed point behavior of these oscillators near resonance. We find a relation between the amplitude frequency coupling of the oscillators and their phase-locking behavior. In particular, we show that two hard oscillators lock in phase and two soft oscillators lock out of phase. We compare our theoretical predictions with computer simulations of two examples: a sinusoidally driven X^3 force oscillator and two coupled van der Pol oscillators with X^3 force.

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I. INTRODUCTION

Weakly coupled nonlinear oscillators have been a popular research topic for many decades [1–7]. A review article by Chirikov [8] shows how the dynamics of a large class of two weakly coupled conservative oscillators near a resonance can be approximated by the dynamics of a pendulum. We study the same systems with the addition of non-Hamiltonian perturbations, such as damping. There is a lot of phenomenological and numerical work done on these systems, for example, by Bohr *et al.* [5]. There are many physical applications of such systems, including Josephson junctions [9–12], driven charge density waves [13], and oscillator models of Karman vortex streets in fluids [14].

In the usual secular perturbation theory [1,2] canonical action-angle variables are introduced, secular terms in the equations of motion are removed with a special change of variables, the Hamiltonian is averaged over one of the fast angle variables, and the Hamiltonian is expanded about the resonance. This reduces the original four-dimensional two-oscillator system to a two-dimensional dynamical system. This averaging is responsible for removing the stochastic dynamics between the Kol'mogorov-Arnol'd-Moser [15] surfaces. Because of the non-Hamiltonian forces such as damping, our method differs in that we average and expand the equations of motion, and we use transformation functions to change to action-angle variables as opposed to the generating functions of Hamiltonian mechanics.

There is a large body of literature concerned with linear oscillators that are perturbed by small nonlinear forces [3,16]. For example, Nayfeh and Mook [3] applied the method of multiple scales to a sinusoidally driven Duffing oscillator. In this paper, we study oscillators that may have large nonlinear forces that are not treated as perturbations. In our analysis, we start with nonlinear conservative oscillators and add coupling (or driving) and non-Hamiltonian perturbations.

In Sec. II, we derive a standard equation of motion for a general class of two coupled oscillators. In Sec. III, we discuss some general properties of the standard equation. In Sec. IV, we apply the results of Sec. II to an example of a sinusoidally driven X^3 force oscillator. In Sec. V, we apply the results of Sec. II to an example of two coupled X^3 force van der Pol oscillators. In the Appendix, we present action-angle transformation functions that are used to derive the standard equation in Sec. II.

II. STANDARD EQUATION

A. Derivation of the standard equation

We investigate the system of two coupled oscillators of the form

$$\dot{x}_i = \frac{p_i}{m_i}, \tag{2.1}$$

$$\dot{p}_i = F_i(x_i) + \epsilon\Gamma_i(x_i,p_i) + \epsilon k_i x_j, \tag{2.2}$$

$$F_i(x_i) = -\frac{dU_i(x_i)}{dx}, \tag{2.3}$$

where $i,j = 1,2$, $i \neq j$, x_i is a displacement, k_i and m_i are constants, ϵ is a small constant number, and U_i is the potential of the ith oscillator. The analysis of this paper is restricted to solutions of Eqs. (2.1) and (2.2) that are bounded in some finite region of x_1, x_2, p_1, and p_2 space for all t. In this section, we look at both the case of a driven oscillator (one-way coupling) and the case of two coupled oscillators with two-way coupling. For the case of a driven oscillator, we set $k_1 \equiv 0$, $k_2 \equiv k$, and $\Gamma_1 \equiv 0$. For the case with two-way coupling, we set $k_1 \equiv k$ and $k_2 \equiv k$.

Next, we change variables x_i and p_i to their corresponding action-angle variables J_i and ϕ_i by using their corresponding transformation functions $X_i(J_i, \phi_i)$, $P_i(J_i, \phi_i)$, $K_i(J_i)$, and $\omega_i(J_i)$ (i=1,2) as explained in the appendix [17].

In what follows, we investigate the dynamics of Eqs. (2.1) and (2.2) close to a primary resonance, which is when the angular frequencies of the two oscillators are equal. In order to avoid secular terms, we use the following change of dynamical variables:

$$\Phi \equiv \phi_2 - \phi_1, \tag{2.4}$$

$$\Omega \equiv \omega_2(J_2) - \omega_1(J_1), \tag{2.5}$$

$$e \equiv e(J_1, J_2), \tag{2.6}$$

$$\phi \equiv \phi_1, \tag{2.7}$$

where the function $e(J_1, J_2)$ is defined by $e \equiv J_1$, for the case of a driven oscillator, and $e \equiv J_2 + J_1$, for the case with two-way coupling.

From Eqs. (A21), (A35), and (A36) from the Appendix and Eqs. (2.1)–(2.7), the equations of motion of Φ, Ω, e, and ϕ may be written as

$$\dot{\Phi} = \Omega - \epsilon \frac{\partial X_2}{\partial J_2} \left(\Gamma_2 + k_2 X_1\right) + \epsilon \frac{\partial X_1}{\partial J_1} \left(\Gamma_1 + k_1 X_2\right), \tag{2.8}$$

$$\dot{\Omega} = \frac{\epsilon}{m_2 \omega_2} \frac{\partial \omega_2}{\partial J_2} P_2 \left(\Gamma_2 + k_2 X_1\right) - \frac{\epsilon}{m_1 \omega_1} \frac{\partial \omega_1}{\partial J_1} P_1 \left(\Gamma_1 + k_1 X_2\right), \tag{2.9}$$

$$\dot{e} = \frac{\epsilon}{m_2 \omega_2} \frac{\partial e}{\partial J_2} P_2 \left(\Gamma_2 + k_2 X_1\right) + \frac{\epsilon}{m_1 \omega_1} \frac{\partial e}{\partial J_1} P_1 \left(\Gamma_1 + k_1 X_2\right), \tag{2.10}$$

$$\dot{\phi} = \omega_1 - \epsilon \frac{\partial X_1}{\partial J_1} \left(\Gamma_1 + k_1 X_2\right), \tag{2.11}$$

where it is assumed that all quantities on the right-hand side of the equations can be written as functions of just Φ, Ω, e, and ϕ. Equations (2.8)–(2.11) are exact.

The two-oscillator system is near a primary resonance, so we impose that Ω is of order ϵ. By Eqs. (2.8)–(2.11), Φ, Ω, and e are changing, by a factor of order ϵ, more slowly than ϕ. Because of this, we approximate the dynamics of Φ, Ω, and e with the dynamics of an averaged version of these equations of motion by averaging over the fast variable ϕ [1,2,6,18,19]. We define an *averaged* function $\langle \xi \rangle_\phi(\Phi, \Omega, e)$ by

$$\langle \xi \rangle_\phi(\Phi, \Omega, e) \equiv \frac{1}{2\pi} \int_0^{2\pi} \xi(\Omega, \Phi, e, \phi)\, d\phi\,. \tag{2.12}$$

In what follows, we use

$$X_i(J_i, \phi_i) = \sum_{l=0}^{\infty} C_{i,l}(J_i) \cos(l\phi_i), \tag{2.13}$$

$$P_i(J_i, \phi_i) = -m_i \omega_i(J_i) \sum_{l=1}^{\infty} l C_{i,l}(J_i) \sin(l\phi_i), \tag{2.14}$$

where i=1,2. We consider the case where the Fourier coefficients $C_{i,l}(J_i)$, as defined in Eqs. (2.13) and (2.14), decrease rapidly with increasing l, so we approximate $C_{i,l}(J_i) = 0$ for $l > 1$ in terms that have a Φ dependence after the averaging. These two approximations, averaging and then dropping harmonics, give

$$\dot{\Phi} = \Omega - \epsilon \left\langle \frac{\partial X_2}{\partial J_2} \Gamma_2 \right\rangle_\phi + \epsilon \left\langle \frac{\partial X_1}{\partial J_1} \Gamma_1 \right\rangle_\phi - \frac{\epsilon}{2} \left(k_2 \frac{dC_{2,1}}{dJ_2} C_{1,1} - k_1 \frac{dC_{1,1}}{dJ_1} C_{2,1} \right) \cos\Phi, \tag{2.15}$$

$$\dot{\Omega} = \frac{\epsilon}{m_2 \omega_2} \frac{\partial \omega_2}{\partial J_2} \langle P_2 \Gamma_2 \rangle_\phi - \frac{\epsilon}{m_1 \omega_1} \frac{\partial \omega_1}{\partial J_1} \langle P_1 \Gamma_1 \rangle_\phi - \frac{\epsilon}{2} C_{1,1} C_{2,1} \left(k_2 \frac{\partial \omega_2}{\partial J_2} + k_1 \frac{\partial \omega_1}{\partial J_1} \right) \sin\Phi, \tag{2.16}$$

$$\dot{e} = \frac{\epsilon}{m_2 \omega_2} \frac{\partial e}{\partial J_2} \langle P_2 \Gamma_2 \rangle_\phi + \frac{\epsilon}{m_1 \omega_1} \frac{\partial e}{\partial J_1} \langle P_1 \Gamma_1 \rangle_\phi, \tag{2.17}$$

where all the terms with $\langle\,\rangle_\phi$ do not depend on Φ.

We expand the right-hand sides of Eqs. (2.15)–(2.17) in a Taylor series about $\Omega = 0$ and about $e = e_0$, where e_0 is a constant. We get the value of e_0 from the condition $\dot{e}(\Omega = 0, e = e_0) = 0$.

Φ is of size of order 1. $\dot{\Phi}$ is expanded to first order in ϵ. We impose that Ω is of size of order ϵ. To be consistent with the expansion of $\dot{\Phi}$, $\dot{\Omega}$ is expanded to order ϵ^2. Due to the condition, $\dot{e}(\Omega = 0, e = e_0) = 0$, the first nonzero term in the expansion of $\dot{e}$ is of size of order ϵ^2. This makes $e - e_0$ of size of order ϵ^2 for t of size of order 1. This, in turn, makes the lowest order term that depends on e in the expansion of $\dot{\Phi}$ and $\dot{\Omega}$ of size of order ϵ^3. So by expanding to second order, the $\dot{\Phi}$ and the $\dot{\Omega}$ equations do not depend on e. The above restrictions and approximations enable us to study a two-dimensional system in the variables Φ and Ω, which evolve independently of e and ϕ.

Expanding Eqs. (2.15) and (2.16) gives

$$\dot{\Phi} = \Omega + a_1 + a_2 \cos\Phi, \tag{2.18}$$

$$\dot{\Omega} = a_3 + a_4 \sin\Phi + a_5 \Omega + a_6 \Omega \sin\Phi, \tag{2.19}$$

where a_i (i= 1,2,3,4,5,6) are constants of size of order ϵ. We call Eqs. (2.18) and (2.19) the standard equation because it describes the dynamics about resonance for a more general class of two coupled oscillators than Lichtenberg and Lieberman's standard Hamiltonian [1].

B. Linear fixed point analysis of the standard equation

An analytic expression for the fixed points of Eqs. (2.18) and (2.19) cannot, in general, be found. We can, however, find analytic expressions for the fixed points to first order in ϵ. To first order, if $a_4 \geq a_3$, there are two fixed points in Eqs. (2.18) and (2.19) with $0 \leq \Phi < 2\pi$: one at

$$\Phi_{f,1} = \left[\sin^{-1}\left(\frac{a_3}{a_4}\right) + \pi\right] \mod 2\pi, \tag{2.20}$$

$$\Omega_{f,1} = -a_1 + a_2\sqrt{1 - \left(\frac{a_3}{a_4}\right)^2} \qquad (2.21)$$

and one at

$$\Phi_{f,2} = \left[-\sin^{-1}\left(\frac{a_3}{a_4}\right)\right] \mod 2\pi, \qquad (2.22)$$

$$\Omega_{f,2} = -a_1 - a_2\sqrt{1 - \left(\frac{a_3}{a_4}\right)^2} \qquad (2.23)$$

where $\Phi_{f,i}$ (i=1,2) is a fixed point value of Φ and $\Omega_{f,i}$ (i=1,2) is a fixed point value of Ω.

To lowest order in ϵ, the characteristic exponents ($\Phi - \Phi_{f,i}, \Omega - \Omega_{f,i} \sim e^{s_i t}$) are given by

$$s_i = \frac{1}{2}\left(\frac{a_3 a_2}{a_4} + a_5 - \frac{a_3 a_6}{a_4}\right) \pm \sqrt{-\frac{a_4}{a_2}\left(\Omega_{f,i} + a_1\right)}, \qquad (2.24)$$

where i = 1,2 and the $\pm$ gives two values of s_i for each i. If $a_4 < a_3$ then, to first order in ϵ, there are no fixed points in the standard equation. If $a_4 > a_3$, there is one saddle point and one focal point. The focal point is of special interest to us because it is the point where the two oscillators can phase lock ($\Phi \to$ const) if it is stable (Re$[s_i] < 0$). There is a bifurcation line at $a_3 = a_4$ where the saddle point and the focus collide and become one fixed point (to lowest order in ϵ).

III. PROPERTIES OF THE STANDARD EQUATION DYNAMICS

A. Hard oscillators lock in phase and soft oscillators lock out of phase

Phase locking occurs at a focal point in Φ-Ω space. A linear fixed-point analysis (Sec. II B) of the standard equation, Eqs. (2.18) and (2.19), with $|a_4| \gg |a_3|$ gives us the result that there is a focal point at the fixed point with $\Phi_{f,1} \approx \pi$ if $a_4 > 0$ and at the point with $\Phi_{f,2} \approx 0$ if $a_4 < 0$:

$$a_4 = -\frac{\epsilon}{2} C_{1,1} C_{2,1}\left(k_1 \frac{\partial \omega_1}{\partial J_1} + k_2 \frac{\partial \omega_2}{\partial J_2}\right)\Bigg|_{\Omega=0, e=e_0}, \qquad (3.1)$$

where all quantities are evaluated at $\Omega = 0$ and $e = e_0$.

We define the nonlinearity parameter by

$$G \equiv \frac{1}{k_2}\left(k_1 \frac{\partial \omega_1}{\partial J_1} + k_2 \frac{\partial \omega_2}{\partial J_2}\right)\Bigg|_{\Omega=0, e=e_0}, \qquad (3.2)$$

where all quantities are evaluated at $\Omega = 0$ and $e = e_0$. For the case of one-way coupling, G is the amplitude-frequency coupling of the driven oscillator, $\frac{\partial \omega_2}{\partial J_2}$. For the case of two-way coupling, G is the sum of the amplitude-frequency coupling of the two oscillators $\frac{\partial \omega_1}{\partial J_1} + \frac{\partial \omega_2}{\partial J_2}$.

Assuming that $C_{1,1}$ and $C_{2,1}$ are both positive, if the sign of G is negative then the focal point is at $\Phi_{f,1} \approx \pi$ and if G is positive then the focal point is at $\Phi_{f,2} \approx 0$. We define an oscillator with a frequency that decreases with increasing action ($\frac{\partial \omega}{\partial J} < 0$) to be a soft oscillator and an oscillator with a frequency that increases with increasing action ($\frac{\partial \omega}{\partial J} > 0$) to be a hard oscillator. This shows that if the coupling forces are larger than other perturbations, i.e., $|a_3| \gg |a_4|$, and if the coupled oscillator system can phase lock, i.e., Re$[s_i] < 0$, then if $G > 0$ (hard oscillators) the oscillators phase lock in phase ($\Phi \to 0$) and if $G < 0$ (soft oscillators) the oscillators phase lock out of phase ($\Phi \to \pi$).

B. Two coupled identical oscillators

We define two identical oscillators to be two oscillators that have the same equations of motion when $k_i = 0$ for i =1,2. Inspection of the derivation of the standard equation, Eqs. (2.18) and (2.19), from Eqs. (2.1) and (2.2) shows that for two identical oscillators the constants a_1,a_2,a_3,a_6 are zero. With this, the standard equation is given by

$$\dot{\Phi} = \Omega, \qquad (3.3)$$

$$\dot{\Omega} = a_4 \sin\Phi + a_5 \Omega, \qquad (3.4)$$

where

$$a_4 = \left[-\epsilon k C_{1,1}^2 \frac{\partial \omega_1}{\partial J_1}\right]\Bigg|_{\Omega=0, e=e_0}, \qquad (3.5)$$

$$a_5 = \left[\frac{\epsilon}{m_1 \frac{\partial \omega_1}{\partial J_1}} \frac{\partial}{\partial J_1}\left(\frac{1}{\omega_1}\frac{\partial \omega_1}{\partial J_1}\langle P_1 \Gamma_1\rangle_\phi\right)\right]\Bigg|_{\Omega=0, e=e_0}, \qquad (3.6)$$

where the expressions for a_4 and a_5 are evaluated at $\Omega = 0$ and $e = e_0$. We see that the dynamics of two coupled identical oscillators with two-way coupling, near resonance, is that of a damped pendulum. So if $a_5 < 0$, the two oscillators phase-lock for all initial conditions in the Φ-Ω plane that are near the primary resonance.

IV. SINUSOIDALLY DRIVEN X^3 FORCE OSCILLATOR

A. Derivation of the standard equation

We now investigate two coupled oscillators with the equation of motion

$$\dot{x}_1 = \frac{p_1}{m_1}, \qquad (4.1)$$

$$\dot{p}_1 = -m_1 \omega_1^2 x_1, \qquad (4.2)$$

$$\dot{x}_2 = \frac{p_2}{m_2}, \qquad (4.3)$$

$$\dot{p}_2 = -c{x_2}^3 + \epsilon k x_1 - \epsilon d p_2, \qquad (4.4)$$

where ω_1, m_1, m_2, c, k, and d are constants and ϵ is a small constant number.

We define the unperturbed forcing functions to be $F_1(x_1) = -m_1\omega_1^2 x_1$ and $F_2(x_2) = -c{x_2}^3$. Note that this coupled oscillator system is of the form of the more general coupled oscillator system, Eqs. (2.1) and (2.2) in Sec. II A, with $k_1 \equiv 0$, $k_2 \equiv k$, $\Gamma_1 \equiv 0$, and $\epsilon\Gamma_2 \equiv -\epsilon d p_2$.

We next change variables x_i and p_i to their corresponding action-angle variables J_i and ϕ_i by using their corresponding transformation functions $X_i(J_i,\phi_i)$, $P_i(J_i,\phi_i)$, $K_i(J_i)$, and $\omega_i(J_i)$ $(i = 1,2)$ as explained in the appendix. This gives

$$C_{1,l}(J_1) = \delta_{1,l}A_1(J_1), \tag{4.5}$$

$$X_1(J_1,\phi_1) = A_1(J_1)\cos\phi_1, \tag{4.6}$$

$$P_1(J_1,\phi_1) = -m_1\omega_1 A_1(J_1)\sin\phi_1, \tag{4.7}$$

$$K_1(J_1) = \frac{1}{2}m_1\omega_1^2[A_1(J_1)]^2, \tag{4.8}$$

$$\omega_1(J_1) = \omega_1, \tag{4.9}$$

$$A_1(J_1) = \sqrt{\frac{2J_1}{m_1\omega_1}}, \tag{4.10}$$

$$C_{2,l}(J_2) = B_l A_2(J_2), \tag{4.11}$$

$$X_2(J_2,\phi_2) = A_2(J_2)\sum_{l=1}^{\infty} B_l \cos l\phi_2, \tag{4.12}$$

$$P_2(J_2,\phi_2) = -m_2\omega_2(J_2)A_2(J_2)\sum_{l=1}^{\infty} lB_l \sin l\phi_2, \tag{4.13}$$

$$K_2(J_2) = \frac{1}{4}c[A_2(J_2)]^4, \tag{4.14}$$

$$\omega_2(J_2) = \rho A_2(J_2), \tag{4.15}$$

$$A_2(J_2) = \sqrt[3]{\frac{3\rho J_2}{c}}, \tag{4.16}$$

$$\rho \equiv \frac{\pi\sqrt{c}}{3\sqrt{2m_2}\int_0^1 \sqrt{1-z^4}dz}, \tag{4.17}$$

where B_l are constants and the functions $A_1(J_1)$ and $A_2(J_2)$ are introduced for convenience. The first four nonzero values of B_l are approximately 0.9550, 0.043 05, 0.001 860, and 8.040×10^{-5} for $l = 1$, 3, 5, and 7, respectively. ρ is approximately $0.8472\sqrt{\frac{c}{m_2}}$.

With Eqs. (4.5)–(4.17), Eqs. (2.15) and (2.16) can be written as

$$\dot{\Phi} = \Omega - \frac{\epsilon k}{2}\frac{B_1\rho A_1}{cA_2(J_2)^2}\cos\Phi, \tag{4.18}$$

$$\dot{\Omega} = -\frac{\epsilon d}{2}\rho^3\frac{m_2}{c}\left(\sum_{l=0}^{\infty} l^2 {B_l}^2\right)A_2(J_2) - \frac{\epsilon k}{2}\frac{\rho^2 B_1 A_1}{cA_2(J_2)}\sin\Phi. \tag{4.19}$$

Expanding the right-hand sides of Eqs. (4.18) and (4.19) as explained in Sec. II A gives

$$\dot{\Phi} = \Omega - \frac{\epsilon k}{2}\frac{\rho^3 B_1 A_1}{c\omega_1^2}\cos\Phi, \tag{4.20}$$

$$\dot{\Omega} = -\frac{\epsilon d}{2}\rho^2\omega_1\frac{m_2}{c}\left(\sum_{l=0}^{\infty} l^2 {B_l}^2\right) - \frac{\epsilon k}{2}\frac{\rho^3 B_1 A_1}{c\omega_1}\sin\Phi - \frac{\epsilon d}{2}\rho^2\frac{m_2}{c}\left(\sum_{l=0}^{\infty} l^2 {B_l}^2\right)\Omega + \frac{\epsilon k}{2}\frac{\rho^3 B_1 A_1}{c\omega_1^2}\Omega\sin\Phi. \tag{4.21}$$

Equations (4.20) and (4.21) are the standard equation for this driven oscillator system. Comparing it to Eqs. (2.18) and (2.19) can determine the value of the constants a_i $(i = 1,2,3,4,5,6)$ for this system.

The location of a stable focal point of the standard equation corresponds to a stable phase-locked solution of the driven oscillator. When there is no stable focal point in the standard equation there is no stable phase-locked solution of the driven oscillator at a *natural* primary resonance; that is, the steady state solution of the driven oscillator has an angular frequency that is not close to its unperturbed angular frequency $\omega_2(J_{2,f})$, where $J_{2,f}$ is the steady state action of the driven oscillator.

B. Dimensionless equations

There are six physical parameters in this driven oscillator system. Scaling the equations of motion by the length scale α and the time scale γ, as defined by

$$\alpha = \omega_1\sqrt{\frac{m_2}{c}}, \tag{4.22}$$

$$\gamma = \frac{1}{\omega_1}, \tag{4.23}$$

can produce dimensionless equations with just two parameters: a dimensionless damping $\tilde{d}$ and a dimensionless coupling constant $\tilde{k}$, given by

$$\tilde{d} = \frac{\epsilon}{2}\rho^2\left(\sum_{l=0}^{\infty} l^2 {B_l}^2\right)\frac{d}{\omega_1}, \tag{4.24}$$

$$\tilde{k} = \frac{\epsilon}{2}\rho^3 B_1\frac{kA_1\sqrt{c}}{\sqrt{m_2^3\omega_1^3}}. \tag{4.25}$$

In any equation related to this driven oscillator system, the corresponding dimensionless equations can be obtained by setting ϵ, m_1, m_2, c, and ω_1 equal to 1, d equal to $\frac{2\tilde{d}}{\rho^2(\sum_{l=0}^{\infty} l^2 {B_l}^2)}$, and k equal to $\frac{2\tilde{k}}{\rho^3 A_1 B_1}$. With this, the standard equation, Eqs. (4.20) and (4.21), can be written as

$$\dot{\Phi} = \Omega - \tilde{k}\cos\Phi, \tag{4.26}$$

$$\dot{\Omega} = -\tilde{d} - \tilde{k}\sin\Phi - \tilde{d}\Omega + \tilde{k}\Omega\sin\Phi. \tag{4.27}$$

C. Comparing with simulation

To show that the dynamics of the standard equation, Eqs. (4.26) and (4.27), is a good representation of the dynamics of the driven oscillator near the primary resonance, we calculate the values of $\Phi(t)$ and $\Omega(t)$ from the values of $x_i(t)$ and $p_i(t)$ $(i = 1,2)$ from a numerical integration of the driven oscillator, Eqs. (4.1)–(4.4), and compare this to values of $\Phi(t)$ and $\Omega(t)$ we get from the standard equation, Eqs. (4.26) and (4.27). There are four initial conditions necessary to integrate the driven oscillator system Eqs. (4.1)–(4.4): two come from the initial

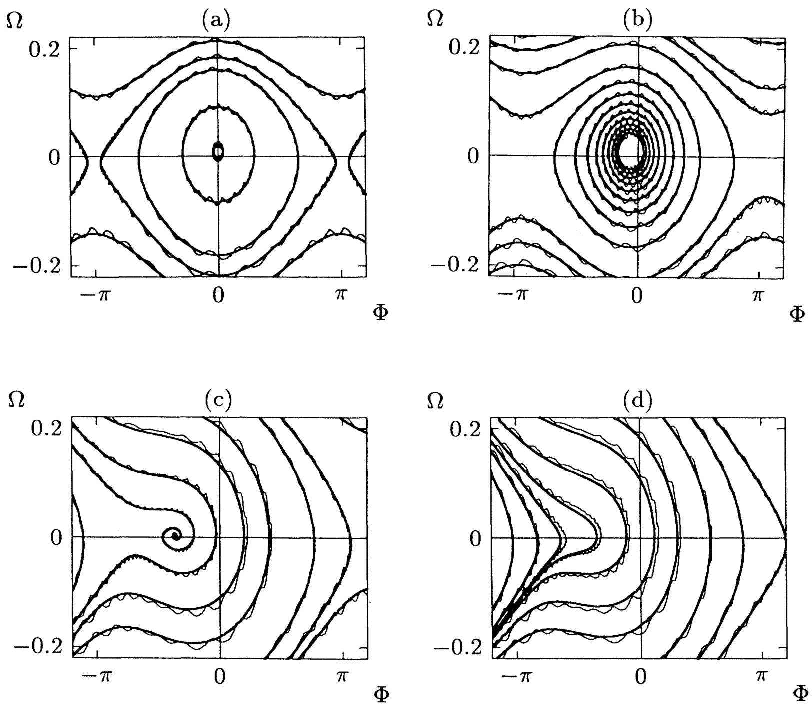

FIG. 1. The Φ-Ω plane by numerically integrating the standard equation, Eqs. (4.26) and (4.27) (thick lines), and by computing Φ and Ω from a numerical integration of the exact equations of motion, Eqs. (4.1)–(4.4) (thin wiggly lines) with (a) $\tilde{k} = 1 \times 10^{-2}$, $\tilde{d} = 0$, and no basin of attraction to the phase-locking (focal) point; (b) $\tilde{k} = 1 \times 10^{-2}$, $\tilde{d} = 0.2 \times 10^{-2}$, and for some initial conditions the dynamics asymptotically approaches the phase-locking (focal) point; (c) $\tilde{k} = 1 \times 10^{-2}$, $\tilde{d} = 0.9 \times 10^{-2}$, and for some initial conditions the dynamics asymptotically approaches the phase-locking (focal) point; and (d) $\tilde{k} = 1 \times 10^{-2}$, $\tilde{d} = 1.1 \times 10^{-2}$, and no fixed points.

values of Φ and Ω, one comes from setting the driver oscillator amplitude A_1 equal to 1, and the last from setting the initial angle (phase) of the driver oscillator. Figure 1 shows Φ-Ω plane plots with $\tilde{k} = 1 \times 10^{-2}$ and several values of $\tilde{d}$.

By comparing Eqs. (2.18) and (2.19) to Eqs. (4.26) and (4.27), we get the values of the a_i ($i = 1,2,3,4,5,6$) constants in the standard equation and use Eqs. (2.22)–(2.24) to get the expressions for the values of the phase-locking point $\Phi_{f,2}$,$\Omega_{f,2}$ and the characteristic exponents s_2 (to lowest order)

$$\Phi_{f,2} = -\sin^{-1}\left(\frac{\tilde{d}}{\tilde{k}}\right), \qquad (4.28)$$

$$\Omega_{f,2} = \sqrt{\tilde{k}^2 - \tilde{d}^2}, \qquad (4.29)$$

$$s_2 = -\frac{3}{2}\tilde{d} \pm \sqrt{-1}\sqrt[4]{\tilde{k}^2 - \tilde{d}^2}. \qquad (4.30)$$

See Fig. 2.

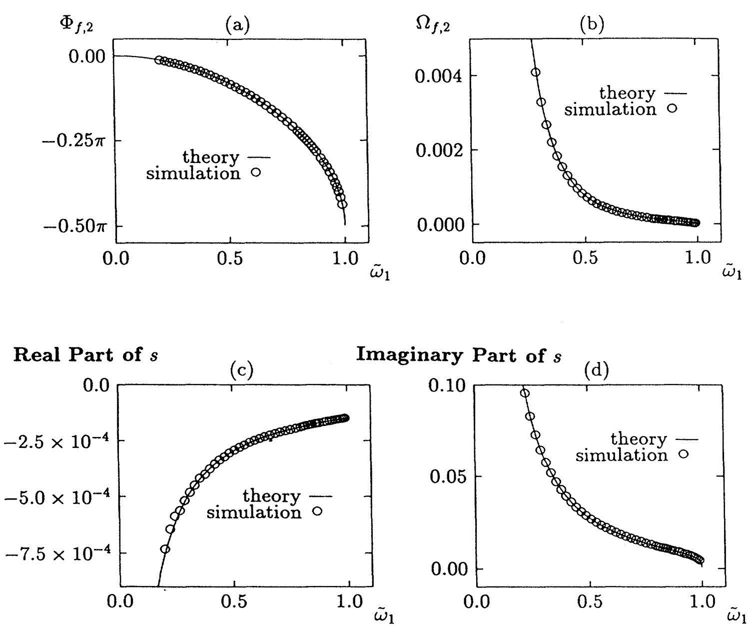

FIG. 2. (a)–(d) With $\tilde{d} = \frac{10^{-4}}{\tilde{\omega}_1}$, $\tilde{k} = \frac{10^{-4}}{\tilde{\omega}_1^3}$, 48 values of $\tilde{\omega}_1$, ranging from 0.2 to 0.99, are used to calculate values of $\Phi_{f,2}$, $\Omega_{f,2}$, and the real and imaginary parts of the characteristic exponents by a numerical simulation of Eqs. (4.1)–(4.4) (simulation) are compared to values from Eq. (4.30) (theory).

V. COUPLED X^3 FORCE VAN DER POL OSCILLATORS

A. Derivation of the standard equation

We now investigate two coupled oscillators with the equation of motion

$$\dot{x}_i = \frac{p_i}{m_i}, \tag{5.1}$$

$$\dot{p}_i = -c_i x_i^3 + \epsilon\eta_i(b_i - x_i^2)p_i + \epsilon k x_j, \tag{5.2}$$

where i,j=1,2, $i \neq j$, m_i, c_i, b_i, η_i, and k are constants, and ϵ is a small constant number. We define the unperturbed forcing functions to be $F_i(x_i) = -c_i x_i^3$. Note that this coupled oscillator system is of the form of Eqs. (2.1) and (2.2) in Sec. II A with $k_i \equiv k$ and $\epsilon\Gamma_i \equiv \epsilon\eta_i(b_i - x_i^2)p_i$.

We next change variables x_i and p_i to their corresponding action-angle variables J_i and ϕ_i by using their corresponding transformation functions $X_i(J_i,\phi_i)$, $P_i(J_i,\phi_i)$, $K_i(J_i)$, and $\omega_i(J_i)$ (i = 1,2) as explained in the Appendix. This gives

$$C_{i,l}(J_i) = B_l A_i(J_i), \tag{5.3}$$

$$X_i(J_i,\phi_i) = A_i(J_i)\sum_{l=1}^{\infty} B_l \cos l\phi_i, \tag{5.4}$$

$$P_i(J_i,\phi_i) = -m_i\omega_i(J_i)A_i(J_i)\sum_{l=1}^{\infty} lB_l \sin l\phi_i, \tag{5.5}$$

$$K_i(J_i) = \frac{1}{4}c_i[A_i(J_i)]^4, \tag{5.6}$$

$$\omega_i(J_i) = \rho_i A_i(J_i), \tag{5.7}$$

$$A_i(J_i) = \sqrt[3]{\frac{3\rho_i J_i}{c_i}}, \tag{5.8}$$

$$\rho_i \equiv \frac{\pi\sqrt{c_i}}{3\sqrt{2m_i}\int_0^1 \sqrt{1-z^4}dz}, \tag{5.9}$$

where i =1,2, B_l are the same constants as in Sec. IV A, and the functions $A_i(J_i)$ are introduced for convenience. ρ_i is approximately $0.8472\sqrt{\frac{c_i}{m_i}}$.

With Eqs. (5.3)–(5.9), Eqs. (2.15)–(2.17) can be written as

$$\dot{\Phi} = \Omega - \frac{\epsilon k}{2}B_1^2\left(\frac{\rho_2}{c_2}\frac{A_1}{A_2^2} - \frac{\rho_1}{c_1}\frac{A_2}{A_1^2}\right)\cos\Phi, \tag{5.10}$$

$$\begin{aligned}\dot{\Omega} = {} & \frac{1}{2}\epsilon\eta_2 b_2\left(\sum_{l=1}^{\infty} l^2B_l^2\right)\frac{m_2}{c_2}\rho_2^3A_2 \\ & -\frac{1}{2}\epsilon\eta_1 b_1\left(\sum_{l=1}^{\infty} l^2B_l^2\right)\frac{m_1}{c_1}\rho_1^3A_1 \\ & -\epsilon\eta_2\sigma\frac{m_2}{c_2}\rho_2^3A_2^3 + \epsilon\eta_1\sigma\frac{m_1}{c_1}\rho_1^3A_1^3 \\ & -\frac{\epsilon k}{2}B_1^2\left(\frac{\rho_2^2}{c_2}\frac{A_1}{A_2} + \frac{\rho_1^2}{c_1}\frac{A_2}{A_1}\right)\sin\Phi,\end{aligned} \tag{5.11}$$

$$\begin{aligned}\dot{e} = {} & \frac{1}{2}\epsilon\eta_2 b_2\left(\sum_{l=1}^{\infty} l^2B_l^2\right)m_2\rho_2A_2^3 \\ & +\frac{1}{2}\epsilon\eta_1 b_1\left(\sum_{l=1}^{\infty} l^2B_l^2\right)m_1\rho_1A_1^3 \\ & -\epsilon\eta_2\sigma m_2\rho_2A_2^5 - \epsilon\eta_1\sigma m_1\rho_1A_1^5,\end{aligned} \tag{5.12}$$

where σ is a constant that is approximately 0.1273. Expanding the right-hand sides of Eqs. (5.10)–(5.12), as explained in Sec. II A, gives

$$\dot{\Phi} = \Omega - \frac{\epsilon k}{2}B_1^2\left(\frac{\rho_2^2}{c_2\rho_1^2} - \frac{\rho_1^2}{c_1\rho_2^2}\right)\frac{\rho_2}{A_{1,0}}\cos\Phi, \tag{5.13}$$

$$\begin{aligned}\dot{\Omega} = {} & \frac{\epsilon}{2}\left(\sum_{l=1}^{\infty} l^2B_l^2\right)\left(\eta_2 b_2\frac{m_2}{c_2}\rho_2^2 - \eta_1 b_1\frac{m_1}{c_1}\rho_1^2\right)\rho_1A_{1,0} \\ & -\epsilon\sigma\left(\eta_2\frac{m_2}{c_2} - \eta_1\frac{m_1}{c_1}\right)\rho_1^3A_{1,0}^3 - \frac{\epsilon k}{2}B_1^2\left(\frac{\rho_2^4}{c_2} + \frac{\rho_1^4}{c_1}\right)\frac{1}{\rho_1\rho_2}\sin\Phi + \frac{\epsilon}{\frac{\rho_2^4}{c_2} + \frac{\rho_1^4}{c_1}} \\ & \times\left[\frac{1}{2}\left(\sum_{l=1}^{\infty} l^2B_l^2\right)\left(\eta_2 b_2\frac{m_2}{c_2^2}\rho_2^6 + \eta_1 b_1\frac{m_1}{c_1^2}\rho_1^6\right) - 3\sigma\left(\eta_2\frac{m_2}{c_2^2}\rho_2^4 + \eta_1\frac{m_1}{c_1^2}\rho_1^4\right)\rho_1^2A_{1,0}^2\right]\Omega \\ & +\frac{\frac{\epsilon k}{2}B_1^2}{\frac{\rho_2^4}{c_2} + \frac{\rho_1^4}{c_1}}\left(\frac{\rho_2^8}{c_2^2} - \frac{\rho_1^8}{c_1^2}\right)\frac{1}{\rho_1^2\rho_2A_{1,0}}\Omega\sin\Phi,\end{aligned} \tag{5.14}$$

$$A_{1,0} \equiv \rho_2\sqrt{\frac{\left(\sum_{l=1}^{\infty} l^2B_l^2\right)}{2\sigma}\left(\frac{\eta_1 b_1 m_1\rho_2^2 + \eta_2 b_2 m_2\rho_1^2}{\eta_1 m_1\rho_2^4 + \eta_2 m_2\rho_1^4}\right)}. \tag{5.15}$$

Equations (5.13) and (5.14) are the standard equation for this driven oscillator system. Comparing it to Eqs. (2.18) and (2.19) determines the value of the constants a_i (i=1,2,3,4,5,6), in the standard equation.

B. Dimensionless equations

There are nine physical parameters in this driven oscillator system. Scaling the equations of motion by the length scale α_1, for oscillator 1, and α_2, for oscillator 2, and the time scale γ, as defined by

$$\alpha_1 = \sqrt{\frac{\left(\sum_{l=1}^{\infty} l^2 B_l^2\right)}{2\sigma} \frac{m_2}{m_1} b_2}, \tag{5.16}$$

$$\alpha_2 = \sqrt{\frac{\left(\sum_{l=1}^{\infty} l^2 B_l^2\right)}{2\sigma} b_2}, \tag{5.17}$$

$$\gamma = \sqrt{\frac{2\sigma}{\left(\sum_{l=1}^{\infty} l^2 B_l^2\right)} \frac{m_2}{c_2 b_2}}, \tag{5.18}$$

can produce dimensionless equations with the five dimensionless parameters

$$\tilde{c}_1 = \frac{c_1}{c_2}\left(\frac{m_2}{m_1}\right)^2, \tag{5.19}$$

$$\tilde{b}_1 = \frac{m_1 b_1}{m_2 b_2}, \tag{5.20}$$

$$\tilde{\eta}_1 = \epsilon\eta_1 \sqrt{\frac{\left(\sum_{l=1}^{\infty} l^2 B_l^2\right)\sigma}{2} \frac{m_2^3 b_2}{m_1^2 c_2}}, \tag{5.21}$$

$$\tilde{\eta}_2 = \epsilon\eta_2 \sqrt{\frac{\left(\sum_{l=1}^{\infty} l^2 B_l^2\right)\sigma}{2} \frac{m_2 b_2}{c_2}}, \tag{5.22}$$

$$\tilde{k} = \frac{\epsilon k B_1^2 \sigma}{\left(\sum_{l=1}^{\infty} l^2 B_l^2\right) b_2 c_2} \sqrt{\frac{m_2}{m_1}}. \tag{5.23}$$

In any equation related to this driven oscillator system, the corresponding dimensionless equations can be obtained by setting m_1, m_2, c_2, and ϵ equal to 1, c_1 equal to $\tilde{c}_1$, b_1 equal to $\frac{2\sigma}{\sum_{l=1}^{\infty} l^2 B_l^2}\tilde{b}_1$, b_2 equal to $\frac{2\sigma}{\sum_{l=1}^{\infty} l^2 B_l^2}$, η_1 equal to $\frac{\tilde{\eta}_1}{\sigma}$, η_2 equal to $\frac{\tilde{\eta}_2}{\sigma}$, and k equal to $\frac{2\tilde{k}}{B_1^2}$. With these dimensionless parameters, the standard equation, Eqs. (5.13) and (5.14), can be written as

$$\dot{\Phi} = \Omega - \tilde{k}\left(1 - \tilde{c}_1\right)\frac{\rho}{\tilde{c}_1 \tilde{A}_{1,0}} \cos\Phi, \tag{5.24}$$

$$\begin{aligned} \dot{\Omega} = {} & \left[\tilde{\eta}_2 - \tilde{\eta}_1\tilde{b}_1 - \left(\tilde{\eta}_2\tilde{c}_1 - \tilde{\eta}_1\right)\tilde{A}_{1,0}^2\right]\sqrt{\tilde{c}_1}\rho^3\tilde{A}_{1,0} \\ & -\tilde{k}\left(1+\tilde{c}_1\right)\frac{\rho^2}{\sqrt{\tilde{c}_1}}\sin\Phi \\ & +\frac{\rho^2}{1+\tilde{c}_1}\left[\tilde{\eta}_2 + \tilde{\eta}_1\tilde{b}_1\tilde{c}_1 - 3\left(\tilde{\eta}_2+\tilde{\eta}_1\right)\tilde{c}_1\tilde{A}_{1,0}^2\right]\Omega \\ & +\tilde{k}\left(1-\tilde{c}_1\right)\frac{\rho}{\tilde{c}_1\tilde{A}_{1,0}}\Omega\sin\Phi, \end{aligned} \tag{5.25}$$

$$\tilde{A}_{1,0} \equiv \sqrt{\frac{\tilde{\eta}_1\tilde{b}_1 + \tilde{\eta}_2\tilde{c}_1}{\tilde{\eta}_1 + \tilde{\eta}_2\tilde{c}_1^2}}, \tag{5.26}$$

$$\rho \equiv \frac{\pi}{3\sqrt{2}\int_0^1 \sqrt{1-z^4}dz}. \tag{5.27}$$

C. Comparing with simulation

To show that the dynamics of the standard equation, Eqs. (5.24) and (5.25), is a good representation of the dynamics of the two coupled oscillators near the primary resonance, we calculate the values of $\Phi(t)$ and $\Omega(t)$ from the values of $x_i(t)$ and $p_i(t)$ (i= 1,2) by numerically integrating the two coupled oscillators, Eqs. (5.1) and (5.2), and compare this to values of $\Phi(t)$ and $\Omega(t)$ that we get by numerically integrating the standard equation, Eqs. (5.24) and (5.25). We display the result of this in Φ-Ω plane plots. There are four initial conditions necessary to integrate the two coupled oscillators: two come from the initial values of Φ and Ω, one comes from $e(t=0) = e_0$, and the last from setting the initial angle (phase) of oscillator one. Figures 3 and 4 show Φ-Ω plane plots for several values of the five dimensionless parameters.

VI. CONCLUSION

We have studied the dynamics of two conservative oscillators with perturbations from a linear displacement coupling and non-Hamiltonian forces such as damping. We have shown that near resonance a large class of driven oscillators and two coupled oscillators can be transformed to the same ordinary differential equations (ODEs) that we call the standard equation: $\dot{\Phi} = \Omega + a_1 + a_2\cos\Phi$, $\dot{\Omega} = a_3 + a_4\sin\Phi + a_5\Omega + a_6\Omega\sin\Phi$, where the constants a_4 and a_5 are weighted sums and a_1, a_2, a_3, and a_6 are weighted differences, which depend on Fourier coefficients and frequencies of the unperturbed oscillators and their derivatives with respect to action. a_2, a_4, and a_6 are directly proportional to the coupling constant. a_1, a_3, and a_5 are directly proportional to other multiplicative constant factors as they appear in the noncoupling perturbing forces (such as damping). The two variables Φ and Ω are defined by $\Phi \equiv \phi_2 - \phi_1$ and $\Omega \equiv \omega_2(J_2) - \omega_1(J_1)$, where J_1, J_2, ϕ_1, and ϕ_2 are defined by action-angle transformation functions, Eqs. (A16) and (A17), and $\omega_2(J_2)$ and $\omega_1(J_1)$ are functionally the same as the angular frequencies of the two uncoupled conservative oscillators.

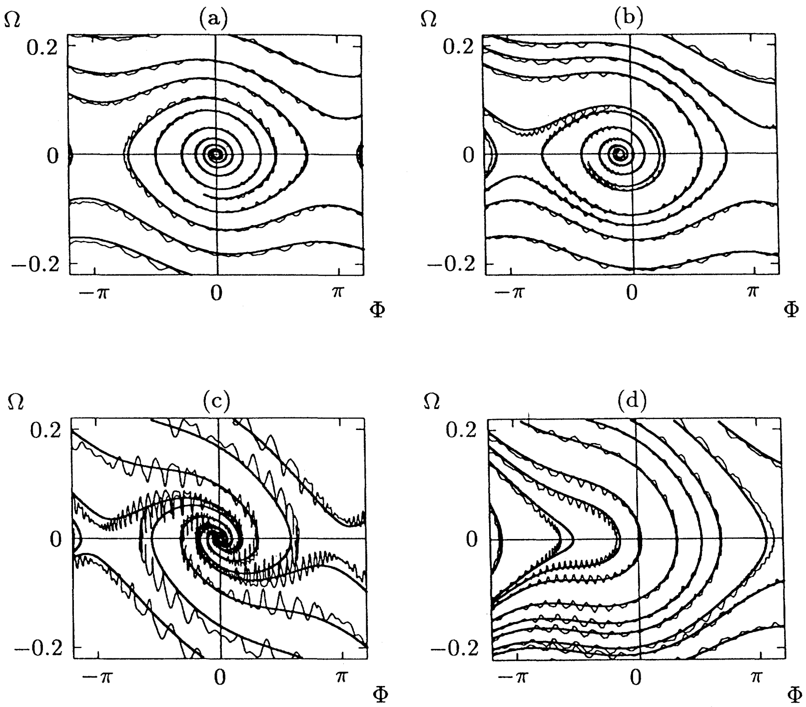

FIG. 3. The Φ-Ω plane by numerically integrating the standard equation, Eqs. (5.24) and (5.25) (thick lines), and by computing Φ and Ω from a numerical integration of the exact equations of motion, Eqs. (5.1) and (5.2) (thin wiggly lines) with (a) $\tilde{k} = 3\times10^{-3}, \tilde{\eta}_1 = \tilde{\eta}_2 = 7\times10^{-3}, \tilde{c}_1 = \tilde{b}_1 = 1$; (b) the same parameters as in (a) except $\tilde{b}_1 = 1.3$; (c) the same parameters as in (a) except $\tilde{\eta}_1 = \tilde{\eta}_2 = 3\times10^{-2}$; and (d) the same parameters as in (a) except $\tilde{b}_1 = 2$; notice that there are no fixed points in this case.

We showed that if $a_4 > a_3$ then there is a focal point and a saddle point in the Φ-Ω space. Then if the real part of the characteristic exponents s_i is less than zero $\frac{1}{2}\left(\frac{a_3a_2}{a_4} + a_5 - \frac{a_3a_6}{a_4}\right) < 0$, then the focal point will be a stable phase-locking point. If, in addition, $a_4 \gg a_3$ (i.e., damping forces are small compared to coupling) the phase difference Φ at the phase-locking point depends only on the sign of the nonlinearity parameter G, where $G \equiv \left(\frac{\partial\omega_1}{\partial J_1} + \frac{\partial\omega_2}{\partial J_2}\right)\Big|_{\Omega=0,e=e_0}$ for two coupled oscillators. This leads to the result that two hard oscillators ($G > 0$) lock in phase and two soft oscillators ($G < 0$) lock out of phase. For the case of two coupled damped identical oscillators we showed that the dynamics near the primary resonance, the standard equation, is that of a damped

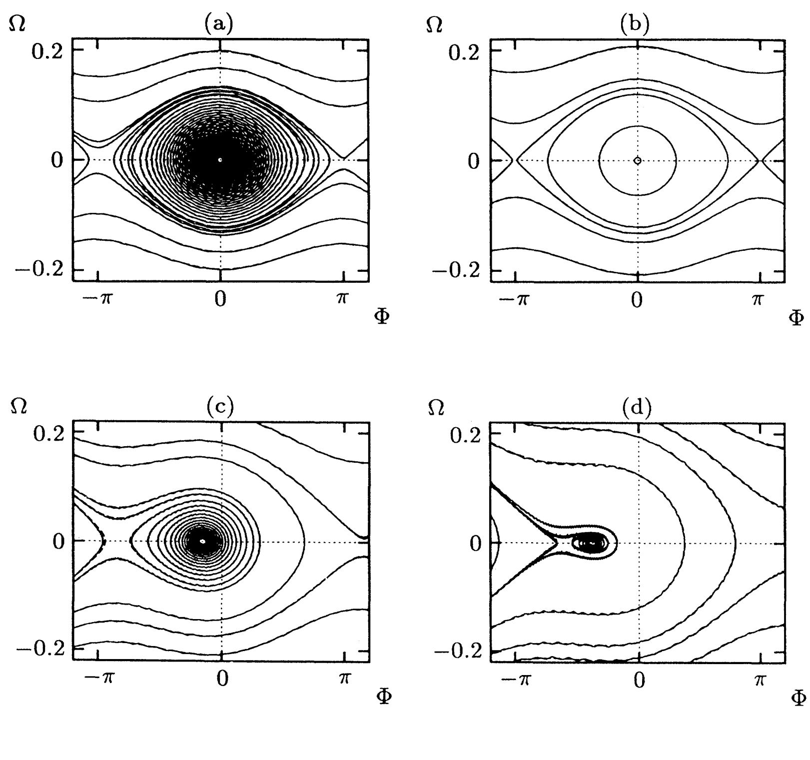

FIG. 4. The Φ-Ω plane by numerically integrating the standard equation, Eqs. (5.24) and (5.25) (dashed lines), and by computing Φ and Ω from a numerical integration of the exact equations of motion, Eqs. (5.1) and (5.2) (continuous wiggly lines), with (a) $\tilde{k} = 3\times10^{-3}, \tilde{\eta}_1 = \tilde{\eta}_2 = 7\times10^{-4}, \tilde{c}_1 = \tilde{b}_1 = 1$; (b) the same parameters as in (a) except $\tilde{\eta}_1 = \tilde{\eta}_2 = 0$; (c) the same parameters as in (a) except $\tilde{b}_1 = 4$; and (d) the same parameters as in (a) except $\tilde{b}_1 = 6$. Note that because of overlap, the dashed lines are not visible at most points.

pendulum $\dot{\Phi} = \Omega$, $\dot{\Omega} = a_4 \sin\Phi + a_5\Omega$.

We have calculated explicit analytic expressions for the constant a_i in the standard equation for two separate oscillator systems, a sinusoidally driven X^3 force oscillator and two coupled van der Pol oscillators with X^3 force. For both systems numerical simulations of the Φ-Ω trajectories were calculated from the original ODEs and the corresponding standard equation. The standard equation's trajectories appeared to be a smoothed shadow of the original ODEs trajectories, even in parameter regions near topographical transitions.

In this paper we have analytically and numerically studied two types of coupled librating oscillators. However, the analytical methods used in this paper (i.e., the transformation functions and averaging method) can be used to study coupled oscillators with other types of coupling forces, coupled rotating oscillators, subharmonic and superharmonic resonances of coupled oscillators, general phase-locking rules of coupled oscillators, and the long-time stability of phase locking of coupled oscillators. For example, by changing the initial conditions in the definitions of the action-angle transformation functions, Eqs. (A19) and (A20), one could study rotating oscillators near resonance. This can be applied to find conditions for phase-locked rotational states in weakly damped coupled Josephson junctions and driven charge density waves. With the change of variables $\Phi \equiv r\phi_1 - s\phi_2$ and $\Omega \equiv r\omega_2 - s\omega_1$, where r and s are integers, it is possible to apply this analysis to subharmonic and superharmonic resonances. In addition, the action-angle transformation functions could be applied to more than two coupled oscillators, such as chains of coupled Josephson junctions [9–12].

In this paper, we averaged the equations of motion over the ϕ variable, Eq. (2.7). This averaging allowed us to ignore the ϕ variable. By averaging and expanding the equation of motion for ϕ, Eq. (2.11), near the resonance, analytic expressions for the angular frequency of the phase-locked oscillators can be derived. These expressions would be just functions of coupling, damping and anti-damping constants, Fourier coefficients and frequencies of the unperturbed oscillators and their derivatives with respect to action evaluated near the resonance. This may lead to general locking rules, such as under what conditions a fast oscillator will lock a slow oscillator, as in some biological and chemical systems [4].

Futher, we ignored the dynamics of the e variable, Eq. (2.6), because it evolved very slowly. The dynamics of e determines the long-time behavior of the coefficients in the standard equation. This could be used for determining when oscillators will lock or unlock at the resonance over long times.

ACKNOWLEDGMENTS

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APPENDIX: ACTION-ANGLE VARIABLES

1. Action-angle transformation functions

Hamiltonian methods cannot be used to solve Eqs. (2.1) and (2.2). We use transformation functions [20], as opposed to generating functions, to change variables to action-angle variables in Sec. II A.

We can use Hamiltonian methods to solve the unperturbed (uncoupled and undamped) single oscillator system

$$\dot{x_0} = \frac{p_0}{m}, \tag{A1}$$

$$\dot{p_0} = F(x_0), \tag{A2}$$

$$F(x_0) = \frac{dU(x_0)}{dx_0}, \tag{A3}$$

where x_0 and p_0 are a pair of canonical variables and U is the potential of the oscillator. We restrict the solutions of Eqs. (A1) and (A2) so that the values of x and p are bound to a finite region of x-p space. Equations (A1) and (A2) can be derived from the Hamiltonian

$$H_0(x_0, p_0) = U(x_0) + \frac{p_0^2}{2m}. \tag{A4}$$

We introduce the action-angle variables [21], J_0 and ϕ_0, of the unperturbed system, by

$$J_0 = \frac{1}{2\pi} \oint p_0 d x_0, \tag{A5}$$

where the integration is carried over a complete cycle of the motion and ϕ_0 is the momentum conjugate of J_0. The equations of motion of J_0 and ϕ_0 are

$$\dot{J_0} = 0, \tag{A6}$$

$$\dot{\phi_0} = \omega_0(J_0), \tag{A7}$$

$$\omega_0(J_0) = \frac{\partial K_0(J_0)}{\partial J_0}, \tag{A8}$$

where $K_0(J_0)$ is the Hamiltonian of the unperturbed system when it is transformed to action-angle variables and $\omega_0(J_0)$ is the angular frequency of the unperturbed system.

We define transformation functions P_0 and X_0, such that

$$p_0 = P_0(J_0, \phi_0), \quad x_0 = X_0(J_0, \phi_0). \tag{A9}$$

From Eqs. (A1), (A2) and (A9)

$$\frac{\partial P_0}{\partial J_0}\dot{J_0} + \frac{\partial P_0}{\partial \phi_0}\dot{\phi_0} = F(X_0), \tag{A10}$$

$$\frac{\partial X_0}{\partial J_0}\dot{J_0} + \frac{\partial X_0}{\partial \phi_0}\dot{\phi_0} = \frac{P_0}{m}. \tag{A11}$$

From Eqs. (A6), (A7), (A10), and (A11)

$$\omega_0(J_0)\frac{\partial P_0}{\partial \phi_0} = F(X_0), \tag{A12}$$

$$\omega_0(J_0)\frac{\partial X_0}{\partial \phi_0} = \frac{P_0}{m}. \tag{A13}$$

Given these transformation functions, the transformed Hamiltonian of the unperturbed system $K_0(J_0)$ is

$$K_0(J_0) = U(X_0) + \frac{P_0^2}{2m}. \tag{A14}$$

Equations (A8) and (A12)–(A14), along with the equation

$$J_0 = \frac{1}{2\pi}\oint P_0 dX_0, \tag{A15}$$

can be used to define the transformation functions $P_0(J_0, \phi_0)$ and $X_0(J_0, \phi_0)$ to within an arbitrary constant.

2. Dynamics of action-angle variables

Now we study the perturbed single oscillator system

$$\dot{x} = \frac{p}{m}, \tag{A16}$$

$$\dot{p} = F(x) + \lambda(x, p, \zeta, t), \tag{A17}$$

$$F(x) \equiv -\frac{dU(x)}{dx}, \tag{A18}$$

where x and p are independent dynamical variables, m is a constant, λ is the perturbation, ζ represents other time-dependent variables that are distinct from p and x, t is time, and U is functionally the same as the function U in Eq. (A3). ζ can contain variables from other systems. The analysis of this paper is restricted to solutions of Eqs. (A16) and (A17) that have values of x and p that are bound to a finite region of x-p space. We do not restrict λ such that it can be derived from a potential. Because of this, we cannot use Hamiltonian methods to solve Eqs. (A16) and (A17).

We transform x and p, in Eqs. (A16) and (A17), to the action-angle variables J and ϕ by using

$$x = X(J, \phi), \tag{A19}$$

$$p = P(J, \phi), \tag{A20}$$

where $X(J, \phi)$ and $P(J, \phi)$ satisfy

$$\omega(J)\frac{\partial X}{\partial \phi} = \frac{P}{m}, \tag{A21}$$

$$\omega(J)\frac{\partial P}{\partial \phi} = F(X), \tag{A22}$$

$$J = \frac{1}{2\pi}\oint P dX, \tag{A23}$$

$$K(J) = U(X) + \frac{P^2}{2m}, \tag{A24}$$

$$\omega(J) = \frac{\partial K(J)}{\partial J}, \tag{A25}$$

with the conditions

$$P(J, 0) = 0, \tag{A26}$$

$$F(X(J, 0)) < 0, \tag{A27}$$

where the integration is carried over a complete cycle of $X(J, \phi)$ and $P(J, \phi)$ as they change when ϕ goes from 0 to 2π with J held constant. The transformation functions X and P are functionally the same as X_0 and P_0, respectively, if the conditions $P_0(J_0, 0) = 0$ and $F(X_0(J_0, 0)) < 0$ are applied. The general solution to Eqs. (A21)–(A27) can be written as

$$X(J, \phi) = \sum_{l=0}^{\infty} C_l(J)\cos(l\phi), \tag{A28}$$

$$P(J, \phi) = -m\omega(J)\sum_{l=1}^{\infty} lC_l(J)\sin(l\phi), \tag{A29}$$

where $C_l(J)$ is a Fourier coefficient that is dependent on J. The Fourier coefficients $C_l(J)$ are the same action-dependent Fourier coefficients as in the solution of the unperturbed oscillator Eqs. (A16) and (A17) with $\lambda = 0$.

We now find the equations of motion for the perturbed system Eqs. (A16) and (A17) in terms of action-angle variables J and ϕ that are defined by Eqs. (A19)–(A27). From Eqs. (A16), (A17), (A19), and (A20)

$$\frac{\partial X}{\partial \phi}\dot{\phi} + \frac{\partial X}{\partial J}\dot{J} = \frac{P}{m}, \tag{A30}$$

$$\frac{\partial P}{\partial \phi}\dot{\phi} + \frac{\partial P}{\partial J}\dot{J} = F(X) + \lambda(X, P, \zeta, t). \tag{A31}$$

Solving the above for $\dot{J}$ and $\dot{\phi}$ gives

$$\dot{\phi} = \frac{\frac{P}{m}\frac{\partial P}{\partial J} - [F(X) + \lambda(X, P, \zeta, t)]\frac{\partial X}{\partial J}}{\frac{\partial X}{\partial \phi}\frac{\partial P}{\partial J} - \frac{\partial X}{\partial J}\frac{\partial P}{\partial \phi}}, \tag{A32}$$

$$\dot{J} = \frac{[F(X) + \lambda(X, J, \zeta, t)]\frac{\partial X}{\partial \phi} - \frac{P}{m}\frac{\partial P}{\partial \phi}}{\frac{\partial X}{\partial \phi}\frac{\partial P}{\partial J} - \frac{\partial X}{\partial J}\frac{\partial P}{\partial \phi}}. \tag{A33}$$

Taking the derivative of Eq. (A24) with respect to J, substituting $F(X)$ for $-\frac{dU(X)}{dX}$, followed by using Eqs. (A21), (A22), and (A25) for substitutions for $F(X)$, $\frac{P}{m}$, and $\frac{\partial K(J)}{\partial J}$, respectively, and rearranging gives

$$\frac{\partial X}{\partial \phi}\frac{\partial P}{\partial J} - \frac{\partial X}{\partial J}\frac{\partial P}{\partial \phi} = 1. \tag{A34}$$

Equation (A34) shows that the denominators in Eqs. (A32) and (A33) simplify to 1. With Eqs. (A21), (A22), and (A32)–(A34), the equation of motion of the perturbed system in the new variables J and ϕ becomes

$$\dot{\phi} = \omega(J) - \frac{\partial X(J, \phi)}{\partial J}\lambda, \tag{A35}$$

$$\dot{J} = \frac{\partial X(J, \phi)}{\partial \phi}\lambda, \tag{A36}$$

where $\lambda = \lambda(X(J, \phi), P(J, \phi), \zeta, t)$. It should be noted that there are no approximations in deriving Eqs. (A35) and (A36).

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[17] Equations (2.1) and (2.2) with $i = 1$ or $i = 2$ are of the form of Eqs. (A16) and (A17), in the Appendix, with $\lambda \equiv \epsilon\Gamma_i(x_i, p_i) + \epsilon k_i x_j$.

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